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On the torque of wavy vortices

By P. M. EAGLES
Department of Mathematics, The City University, London

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Numerical calculations are made of the torque required to sustain a wavy-vortex flow between rotating cylinders. The results are found to agree well with experimental work of Donnelly (1958), and give further confirmation of the validity of Davey, DiPrima & Stuart's (1968) analysis.

1. Introduction

Davey (1962), following Stuart (1958), calculated the torque required to sustain a Taylor-vortex motion in flow between rotating cylinders. This work was confirmed by the more rigorous analysis of Kirchgässner & Sorger (1969).

Let η be the ratio of the radii of the cylinders. Davey's calculations showed good agreement with the experimental results of Donnelly (1958) for the case when the outer cylinder is fixed and $\eta = \frac{1}{2}$, but not such good agreement for $\eta = 0{\cdot}95$. Davey suggested that in the latter case the lack of agreement occurs because the Taylor vortices become modified and develop into a wavy-vortex form of flow at quite moderate speeds of rotation when η is close to unity. The present calculations were undertaken to see whether or not Davey, DiPrima & Stuart's (1968) theory of wavy-vortex flow gives torque values in agreement with the experiments.

We give a very brief summary of the situation below. The reader is referred to Davey *et al.* for a much more extensive review.

Consider two concentric circular cylinders, the outer one fixed and the annulus filled with liquid. We define a Taylor number T proportional to the square of the angular velocity of the inner cylinder. If T is gradually increased experiments indicate that for a given value of η the flow changes at $T = T_c$ from the purely circumferential laminar Couette flow to Taylor-vortex flow, which consists of the Couette flow with superimposed toroidal regularly spaced vortices. At a still higher value of T, above another critical Taylor number T_1, the vortices become modified by a waviness in the azimuthal direction. Both the number of complete waves and the value of T_1 depend upon the value of η. When η is close to unity T_1 is close to T_c numerically (but $T_1 \nrightarrow T_c$ as $\eta \rightarrow 1$). By T_1 we mean the value of T at which waviness is first seen experimentally.

Davey *et al.* introduced a method of analysis which allows us to examine the linear stability of the Taylor-vortex flow and to calculate the consequent development of wavy-vortex flow. In this method one assumes a definite mode of non-axisymmetric disturbance to the Taylor-vortex flow. This mode has m complete waves in the azimuthal direction. One can then calculate the critical Taylor number $T'(m)$ at which the Taylor vortices become linearly unstable. Davey *et al.*

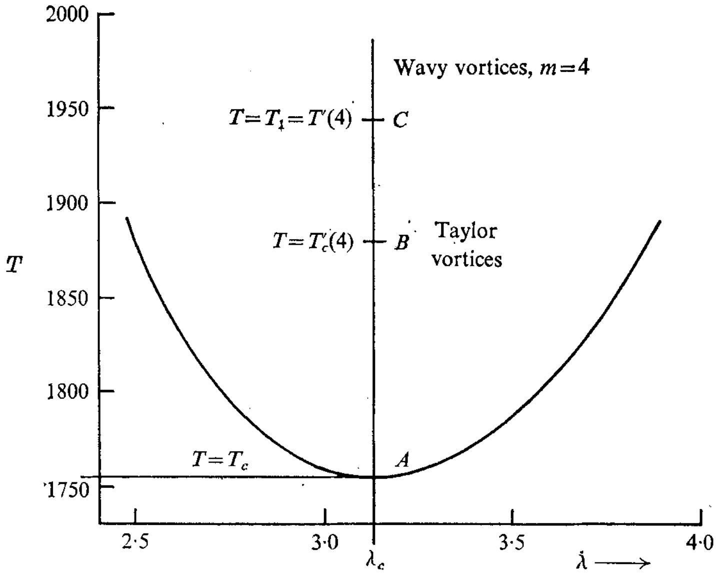

FIGURE 1. The neutral curve for axisymmetric disturbances (2.1) with η = 0·951. We show the Taylor-vortex regime (AB) and the wavy-vortex regime (above C) for $\lambda = \lambda_c$, $m = 4$. The value of $b_{0r}(4)$ is negative on AB and positive above B.

calculated $T'(1)$, $T'(2)$ and $T'(4)$ using the small-gap approximation. Eagles (1971) extended the work. The result is that the $m = 1$ mode is the most unstable, but $T'(1)$, $T'(2)$ and $T'(4)$ are all fairly close together, at about $1{\cdot}1T_c$ for $\eta = 0{\cdot}95$. The tentative conclusion was that $T_1 = T'(1) \simeq 1{\cdot}1T_c$, in qualitative agreement with experiments of Schwarz, Springett & Donnelly (1964). However, the calculations outlined below seem to indicate that $T_1 = T'(4)$.

The present work was undertaken to calculate the torque due to the wavy-vortex model for m = 1, 2, 3 and 4, with $\eta = 0{\cdot}951$,† the main object being to compare the results with Donnelly's (1958) experiments. These experiments show that if the total torque needed to sustain the motion is plotted versus the Taylor number the *rate of increase* of the torque with T shows a definite increase at $T = T_c$, but a definite decrease at a slightly higher value of T, which we tentatively identify with T_1. It is surmized that the flow for $T > T_1$ is of the wavy-vortex type predicted by the method of Davey *et al.* The curve for $T > T_1$ is certainly well below the calculated torque for Taylor vortices (see figure 2).

It is found that for $m = 4$ the results agree well with the experiments of Donnelly (1958) and of Debler, Füner & Schaaf (1968). Although the mode with $m = 1$ is the most unstable theoretically, it appears likely that the mode with $m = 4$ is the one which actually grows to develop into wavy-vortex flow. This may possibly be explained by the higher linear growth rate of the $m = 4$ mode which is predicted by the theory at quite small values of $(T-T_1)/T_c$.

We should also explain at this point that our calculation, like those of Davey and of Davey *et al.*, is restricted to the approximation in which the cubic-terms in the amplitude functions are the highest considered.

† Corresponds to $\delta = 2(R_2-R_1)/(R_2+R_1) = 0{\cdot}05$.

2. The method of expansion

Consider two infinitely long, concentric, right circular cylinders, of radii R_1 and R_2 respectively. We suppose that (r, θ, z) and t are dimensionless cylindrical polar co-ordinates and a dimensionless time. In later formulae and calculations we have used the reference length $R_2 - R_1 = d$ and the reference time d^2/ν, where ν is the kinematic viscosity of the liquid which fills the annulus. The z axis is fixed and coincides with the common axis of the cylinders. The inner cylinder rotates with angular velocity Ω_1 and the outer cylinder is at rest.

There is a steady-state Couette-flow solution of the Navier–Stokes and continuity equations in which the azimuthal (only non-zero) velocity component takes the form $Ar + B/r$, where A and B are constants.

Let u, v and w be dimensionless velocity components of a perturbation to the steady Couette flow. The exact equations for these disturbance velocities are coupled nonlinear partial differential equations with homogeneous boundary conditions. As a first approximation we may linearize. There exist axisymmetric eigenfunctions in which the first velocity component is

$$u = u_1(r)^{\cos}_{\sin}\lambda z\, e^{a_0 t}. \tag{2.1}$$

Calculations show that the eigenvalues are real, and here a_0 is the greatest of these. The wavenumber λ may be specified and then a_0 can be calculated in terms of λ and the other dimensionless parameters which specify the problem. These are the ratio

$$\eta = R_1/R_2 \tag{2.2}$$

and the Taylor number

$$T = 2\eta^2\Omega_1^2 d^4/(1-\eta^2)\,\nu^2. \tag{2.3}$$

There also exist non-axisymmetric linear eigenfunctions in which

$$u = u_2(r)^{\cos}_{\sin}\lambda z\, e^{b_0 t}\, e^{im\theta}. \tag{2.4}$$

Here $u_2(r)$ and b_0 are complex, and b_0 is chosen as the eigenvalue of greatest real part. Then b_0 is a function of λ, m, η and T. We shall write $b_0(m)$ when we wish to emphasize its dependence upon the integer m.

For a given η we may calculate the neutral curve in the λ, T plane, on which $a_0 = 0$. The point $A(\lambda_c, T_c)$ is the minimum (see figure 1). At (λ_c, T_c) it has been calculated that the real part of $b_0(m)$ is negative, at least for $m = 1, 2, 3$ and 4. But if we increase T only slightly we find that the real part of b_0 becomes positive for $T > T_c'(m)$, so that non-axisymmetric modes may be expected to become important.

Davey *et al.* developed an expansion of the nonlinear problem based on the linear eigenfunctions (2.3) and (2.4). The case which yields the wavy vortices is based on (2.3) with $\cos\lambda z$ and (2.4) with $\sin\lambda z$. The expansion uses a real amplitude function $A(t)$ and a complex one $B(t)$. The disturbance velocities are expanded as follows:

$$\left.\begin{aligned} u(r,\theta,z,t) &= A(t)\,u_1(r)\cos\lambda z + B(t)\,u_2(r)\sin\lambda z\, e^{im\theta} + \tilde{B}(t)\,\tilde{u}_2(r)\sin\lambda z\, e^{-im\theta} \\ &\qquad + \sum_{i+j+k>1} A^i(t)\,B^j(t)\,\tilde{B}^k(t)\,u_{ijk}(r,\theta,z), \\ v(r,\theta,z,t) &= A(t)\,v_1(r)\cos\lambda z + B(t)\,v_2(r)\sin\lambda z\, e^{im\theta} + \dots, \\ w(r,\theta,z,t) &= A(t)\,w_1(r)\sin\lambda z + B(t)\,w_2(r)\cos\lambda z\, e^{im\theta} + \dots. \end{aligned}\right\} \tag{2.5}$$

Here a tilde denotes the complex conjugate. In this expansion λ is held fixed at λ_c and m is chosen to be an integer.

The amplitude equations

$$dA/dt = a_0 A + a_1 A^3 + a_4 AB\tilde{B} + \ldots, \tag{2.6}$$

$$dB/dt = b_0 B + b_1 B^2\tilde{B} + b_4 A^2 B + \ldots \tag{2.7}$$

are also needed to ensure consistency of the expansion. It is found that the forms of the functions $u_{ijk}(r,\theta,z)$ and so on are forced by the nonlinear terms. For example,

$$u_{300} = f(r)\cos 3\lambda z + g(r)\cos\lambda z, \tag{2.8}$$

where $f(r)$ and $g(r)$ satisfy certain linear non-homogeneous ordinary differential equations with homogeneous boundary conditions. Further details of the expansion may be found in Davey *et al.* The matrix form of Eagles (1971) was used in the actual calculations, and the modifications needed for the present work are outlined in the appendix.

The constants a_1, a_4, b_1 and b_4 are obtained by using solvability conditions on some of the ordinary differential equations for functions of r, as explained in the appendix.

In the calculations we fix λ, m and η, so that the flow depends only on T. To second order in the amplitudes the flow is obtained by using values of a_1, a_4, b_1 and b_4 which do not vary with T, and by calculating the velocity functions of r like $u_1(r)$, etc., at $T = T_c$. In this approximation the variation of the flow with T is expressed solely through the variation of a_0 and b_0 with T in (2.6) and (2.8). In adapting this approximation we are following Davey (1962) and Davey *et al.* (1968), who explain the idea more fully.

3. Calculation of the torque

Ignoring terms of higher than cubic order in the amplitudes, we find that (2.6) and (2.7) have the steady Taylor-vortex solution

$$B = 0, \quad A = A_0 = (-a_0/a_1)^{\frac{1}{2}} \tag{3.1}$$

for $T > T_c$ $(a_0 > 0)$ provided $a_1 < 0$. The equations also admit the wavy-vortex solution

$$A = A_e, \quad B = \beta_e e^{i\omega t}, \tag{3.2}$$

where the constants A_e, β_e and ω are given by

$$A_e^2 = \frac{a_0 b_{1r} - a_4 b_{0r}}{a_4 b_{4r} - a_1 b_{1r}}, \quad \beta_e^2 = \frac{a_1 b_{0r} - a_0 b_{4r}}{a_4 b_{4r} - a_1 b_{1r}}, \dagger \qquad (3.3), (3.4)$$

$$\omega = b_{0i} + b_{1i}\beta_e^2 + b_{4i}A_e^2. \tag{3.5}$$

The wavy-vortex solution exists only if

$$a_1 b_{0r} - a_0 b_{4r}, \quad a_4 b_{4r} - a_1 b_{1r} \quad \text{and} \quad a_0 b_{1r} - a_4 b_{1r}$$

all have the same sign. Calculations with $\eta = 0{\cdot}951$ show that $a_1 b_{0r} - a_0 b_{4r}$ changes sign from negative to positive at $T = T'(m)$. This is the value of T at

† We use the notation b_{1r} and b_{1i} for the real and imaginary parts of b_1, and so on.

which the Taylor-vortex flow becomes linearly unstable to perturbations of the non-axisymmetric type with m waves (see Davey *et al.* for details). Numerical calculations show that

$$T'(1) = 1924, \quad T'(2) = 1928, \quad T'(3) = 1935 \quad \text{and} \quad T'(4) = 1945.$$

These values differ slightly from those in Eagles (1971) because here we are working to a lower order in the amplitudes. We note for comparison that $T_c = 1753$.

For $T > T'(m)$ calculations show that the wavy-vortex model flow exists. The expansion of the azimuthal velocity v contains terms $A_e^2 F_1(r)$ and $\beta_e^2 F_3(r)$ and the torque required to sustain the wavy-vortex motion can easily be shown to be

$$G = KR|-8/(1+\eta)^2 + A_e^2 F_1'(R_1) + \beta_e^2 F_3'(R_1)|, \tag{3.6}$$

where
$$K = \pi h \rho \nu^2 (1+\eta)\,\eta/2(1-\eta)^2 \tag{3.7}$$

and
$$R = \Omega_1 R_1 (R_2 - R_1)/\nu = [(1+\eta)/2(1-\eta)]^{\frac{1}{2}} T^{\frac{1}{2}}, \tag{3.8}$$

while h is the cylinder length, ρ is the fluid density and ν is the kinematic viscosity. For formula (3.6) the reference length for r is $R_2 - R_1$ and the reference velocity for v is $\frac{1}{4}(R_1 + R_2)\,\Omega_1$.

Formula (3.6) holds only when $T > T'(m)$. For $T < T'(m)$ the flow is of the Taylor-vortex type, and the correct expression for the torque is obtained from (3.6) by setting $B_e^2 = 0$ and replacing A_e^2 by $A_0^2 = -a_0/a_1$.

Equation (3.6) may be re-arranged as

$$G = KR|g_0 + g_1 + g_2|, \tag{3.9}$$

where
$$g_0 = -8/(1+\eta)^2, \tag{3.10}$$

$$g_1 = F_1'(R_1)\,(-a_0/a_1), \tag{3.11}$$

$$g_2 = \{F_3'(R_1) - (a_4/a_1)\,F_1'(R_1)\}\,\beta_e^2. \tag{3.12}$$

Then KRg_0 is the torque associated with the laminar Couette flow, KRg_1 is the extra torque which would be due to the superimposed Taylor-vortex flow if that flow existed, while KRg_2 is the extra torque due to the flow being of the wavy-vortex type.

We have calculated the torque for the case when

$$\eta = 0{\cdot}951, \quad \lambda = 3{\cdot}127 = \lambda_c \quad \text{for} \quad m = 1, 2, 3 \text{ and } 4$$

in order to make a comparison with the experimental results of Donnelly (1958) and of Debler *et al.* (1968). In plotting this figure we have followed Davey (1962), who, in effect, replaced a_0 in (3.11) by the formula

$$a_0 = T_c (da_0/dT)_{T_c} (1 - T_c/T). \tag{3.13}$$

This has the same slope as $T \to T_c$, and has the advantage of fitting experimental results for the torque of Taylor vortices very well for a large range of T above T_c in the case $\eta = 0{\cdot}5$. We found that this procedure makes our results for the total torque with $m = 4$ fit the experimental results of Donnelly (1958) very well

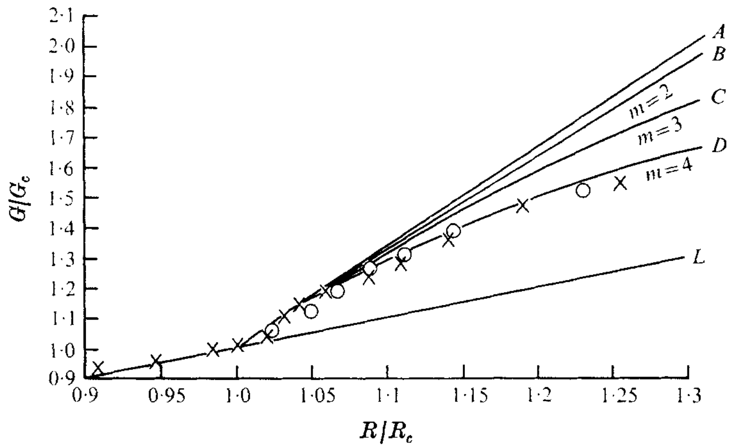

FIGURE 2. The total torque for $\eta = 0{\cdot}95$. Curve L shows the laminar Couette torque. Curve A shows the total torque with Taylor vortices. Curves B, C and D show the total torque with wavy vortices for $m = 2$, 3 and 4. Experimental points: ×, Donnelly & Simon (1960); ○, Debler *et al.* (1968). R is defined in (3.8) and is proportional to $T^{\frac{1}{2}}$.

(figure 2). However, the detailed agreement is probably fortuitous. The important point is that the relative slopes of g_0, g_1 and g_2 agree quite well with the experimental results in the neighbourhood of the transition points T_c and T_1. No more can be expected from our quite crude approximation.

Calculations show that for a considerable range of T around T_c the variation of a_0 and $b_{0r}(m)$ is nearly linear in T. See, for example, Davey *et al.* We therefore used a linear approximation in T to evaluate g_2. Some values of a_0 and of b_{0r} for $m = 1, 2$ and 4 and for several values of T are listed in Eagles (1971). For $m = 3$ we calculated the following values. At $T = 1933$, $b_{0r} = 0{\cdot}7765$; at $T = 1963$, $b_{0r} = 0{\cdot}9844$. We constructed the linear approximations using these figures.

The values of the constants a_1, b_{1r}, a_4, b_{4r}, $F_1'(R_1)$ and $F_3'(R_1)$ are given in table 1. These constants depend upon the scaling adopted for the linear eigenfunctions. In our calculations we took

$$v_1'(R_1) = -iv_2'(R_1) = 2{\cdot}0$$

(equivalent to the second component of $\mathbf{u}_1(-\frac{1}{2})$ and $\mathbf{u}_2(-\frac{1}{2})$ being equal to 1·0 in the notation of Eagles (1971)). It should be noted that our eigenfunction scalings are different from those of Davey *et al.*, so that the numbers in this table are not directly comparable with theirs.

The method of numerical calculation was described in detail in Eagles (1971), where the constants a_1 and b_4 were calculated. We were able to make a further check on the operation of our computer program by making a comparison with Grannick's (1968) calculation of b_1 for the cases $\mu = -0{\cdot}8$, $m = 3$ and 4.† After allowing for the different scalings our results agreed to within less than 1 %, and considering the complexity of the algebra and the different formulations of the numerical problem we consider the agreement to be satisfactory.

In figure 2 we plot the total torque versus the Reynolds number, defined in

† Here μ is the ratio Ω_2/Ω_1, where Ω_2 is the angular velocity of the outer cylinder.

m	a_1	b_{4r}	a_4	b_{1r}	$F_1'(R_1)$	$F_3'(R_1)$
1	−5·53	−5·27	−11·6	−16·8	−1·23	−2·47
2	−5·53	−4·50	−13·2	−17·2	−1·23	−2·49
3	−5·53	−3·30	−15·6	−17·8	−1·23	−2·52
4	−5·53	−1·74	−18·5	−17·9	−1·23	−2·56

TABLE 1

(3.8), in dimensionless form. The constants R_c and G_c are the values of R and G at the first critical point and in plotting the experimental results we estimated R_c and G_c by eye from the detailed experimental points (Donnelly & Simon 1960).

We show the results for $m = 2$, 3 and 4. The curve for $m = 1$ is too close to the Taylor-vortex curve to show. We wish to note that if we do not use Davey's modification (3.13) the agreement with experiment is not so good. If we use the linear approximation in T for a_0 in calculating the Taylor-vortex torque g_1 we find that at $R/R_c = 1{\cdot}2$ the theoretical torque values are increased by about 10 % over those shown in figure 2.

4. Conclusions

The results show that the experimental results for the torque agree quite well with the theory of Davey *et al.* (1968) of wavy-vortex flow with $m = 4$ (four azimuthal waves). It can be pointed out, however, that the $m = 4$ mode of linear disturbance to Taylor-vortex flow is not the theoretically most unstable. As T is gradually increased the $m = 1$ mode appears first (theoretically) at $T = T'(1)$. For a more complete treatment of the problem we would need to consider the interaction of the Taylor-vortex flow with the $m = 1, 2, 3$ and 4 and higher modes. Nevertheless, the critical values $T'(1)$, $T'(2)$, $T'(3)$ and $T'(4)$ are close together (and indistinguishable on figure 2). Also the linear growth rate of the $m = 4$ mode is greatest for quite moderate values of T. Extrapolating from table 3 of Eagles (1971), we estimate that in the linear approximation to B, $(dB/dt)/B$ equals 0·01, 0·07 and 0·11 for $m = 1$, 2 and 4 respectively at $T = 2200$. Thus the $m = 4$ mode might dominate the other modes under some circumstances and grow to produce the wavy-vortex flow with four complete waves. On the other hand it is possible that some combination of the $m = 1$, 2, 3 and 4 modes produces a complex flow with a torque similar to that of our calculation for $m = 4$. Donnelly (1958) made no observations of the azimuthal periodicity.

It has been suggested in the literature that the realized flow for given T could be obtained by maximizing some quantity, e.g. the torque. The present result, showing that the more complex wavy-vortex flow has a smaller torque than the Taylor-vortex flow, indicates that this idea is not correct for the torque at any rate.

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Appendix

The matrix method of formulating the double amplitude expansion of Davey *et al.* has been described in Eagles (1971). The constants a_1 and b_4 were discussed in that paper for the case of an expansion to fifth order in the amplitudes. Here we are interested only in a third-order expansion, but we need to find the extra constants b_1 and a_4.

We use the notation of Eagles (1971) except that here

$$\mathbf{u}_{lmn}^{(p,q)} \text{ is the coefficient of } A^l \tilde{B}^m B^n e^{ip\lambda\zeta + iqk\phi} \tag{A 1}$$

in the expansion of the velocity vector. The co-ordinate ζ is the non-dimensional form of z, and ϕ is a multiple of θ. We find that the linear eigenfunction for the axisymmetric case satisfies the equation

$$(d/dx - \mathbf{A}^{(1,0)} - a_0\mathbf{B})\,\mathbf{u}_{100}^{(1,0)} = 0, \tag{A 2}$$

and the linear eigenfunction for the non-axisymmetric case satisfies the equation

$$(d/dx - \mathbf{A}^{(1,1)} - b_0\mathbf{B})\,\mathbf{u}_{010}^{(1,1)} = 0. \tag{A 3}$$

Here $\mathbf{A}^{(p,q)}$ is defined in Eagles (1971), and is a 6×6 matrix function of x (the dimensionless form of r) and of the dimensionless parameters T, η, μ, λ and m. The matrix $\mathbf{B}$ is constant. The boundary conditions for (A 2) and (A 3) and for the subsequent equations are that the last three components of the six-vector be zero at the cylindrical boundaries ($x = \pm\frac{1}{2}$).

Proceeding, we find eventually that

$$\{d/dx - \mathbf{A}^{(1,0)} - 3a_0\mathbf{B}\}\,\mathbf{u}_{300}^{(1,0)} = \mathbf{N}_8 + a_1\mathbf{B}\mathbf{u}_{100}^{(1,0)}, \tag{A 4}$$

$$\{d/dx - \mathbf{A}^{(1,1)} - (2a_0 + b_0)\,\mathbf{B}\}\,\mathbf{u}_{210}^{(1,1)} = \mathbf{N}_{10} + b_4\mathbf{B}\mathbf{u}_{010}^{(1,0)}, \tag{A 5}$$

$$\{d/dx - \mathbf{A}^{(1,0)} - (a_0 + b_0 + \tilde{b}_0)\,\mathbf{B}\}\,\mathbf{u}_{111}^{(1,0)} = \mathbf{M}_1 + a_4\mathbf{B}\mathbf{u}_{100}^{(1,0)}, \tag{A 6}$$

$$\{d/dx - \mathbf{A}^{(1,1)} - (2b_0 + \tilde{b}_0)\,\mathbf{B}\}\,\mathbf{u}_{021}^{(1,1)} = \mathbf{M}_2 + b_1\mathbf{B}\mathbf{u}_{010}^{(1,1)}. \tag{A 7}$$

The vectors $\mathbf{N}_8$ and $\mathbf{N}_{10}$ have components which are quadratic functions of components of earlier terms in the series, and explicit forms are given in Eagles (1971). The vectors $\mathbf{M}_1$ and $\mathbf{M}_2$ are easily calculated in a similar way.

We fix the parameters η, μ, λ and m. Then a_0 and b_0 are determined as functions of T through (A 2) and (A 3). We now note that when $a_0 = 0$ the operators on the left-hand sides of (A 2) and (A 4) are identical. Hence, when $a_0 = 0$ (A 4) has a solution satisfying the appropriate boundary conditions if and only if the constant a_1 is chosen to make the right-hand side orthogonal to the adjoint eigenfunction $\mathbf{f}_0$ of (A 2). Hence at $a_0 = 0$

$$a_1 = \left[-\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f}_0 . \mathbf{N}_8\,dx \Big/ \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f}_0 . \mathbf{B}\mathbf{u}_{100}^{(1,0)}\,dx\right]_{a_0=0}. \tag{A 8}$$

Let $\mathbf{g}_0$ be the adjoint eigenfunction to (A 3). Then at $a_0 = 0$ we must have

$$b_4 = \left[-\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{g}_0 \,.\, \mathbf{N}_{10}\, dx \Big/ \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{g}_0 \,.\, \mathbf{B}\mathbf{u}_{010}^{(1,0)}\, dx \right]_{a_0=0}. \quad \text{(A 9)}$$

Now consider (A 6) when $b_{0r} = 0$. Since (A 2) has a solution at the corresponding value of a_0 then at $b_{0r} = 0$ (point B in figure 1) we must have

$$a_4 = \left[-\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f}_0 \,.\, \mathbf{M}_1\, dx \Big/ \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f}_0 \,.\, \mathbf{B}\mathbf{u}_{100}^{(1,0)}\, dx \right]_{b_{0r}=0}. \quad \text{(A 10)}$$

Similarly from (A 7), at $b_{0r} = 0$ we must have

$$b_1 = \left[-\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{g}_0 \,.\, \mathbf{M}_2\, dx \Big/ \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{g}_0 \,.\, \mathbf{B}\mathbf{u}_{010}^{(1,1)}\, dx \right]_{b_{0r}=0}. \quad \text{(A 11)}$$

Davey *et al.* have argued that if we wish to approximate the flow to second order in the amplitudes there is no better choice of the constants than those given above. In calculating the flow field and the torque we therefore calculate a_1, b_4, b_1 and a_4 as above, and calculate the $u_{ijk}^{(l,m)}$ for $i+j+k \leqslant 2$ at $a_0 = 0$ $(T = T_c)$. For the Taylor-vortex and wavy-vortex flows the variation with T is therefore expressed solely through the variation of a_0 and b_0 with T.

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